**MECHANICAL ENGINEERING DEPARTMENT**
**UNITED STATES NAVAL ACADEMY**

# EM423 - INTRODUCTION TO MECHANICAL VIBRATIONS

## CONTINUOUS SYSTEMS - FLEXURAL VIBRATION OF STRINGS

### SYMBOLS

| | |
|---|---|
| $r$ | mass density |
| $A_X$ | Cross-sectional area |
| $T$ | Tension |
| $y$ | Lateral deflection from equilibrium position |
| $x$ | Distance along the string |
| $q$ | Slope of string relative to the x-axis |

### INTRODUCTION

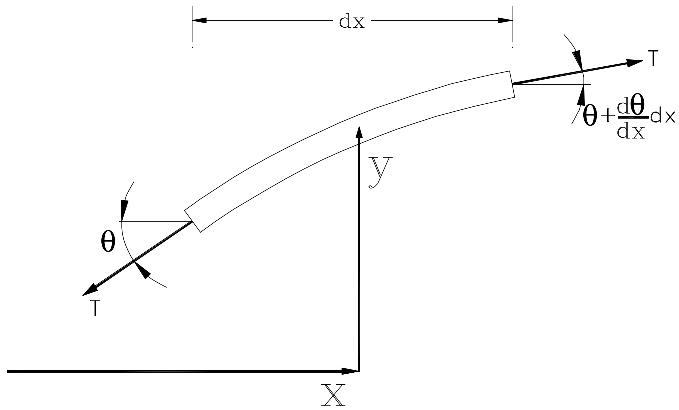
This theory is applicable to tensioned strings undergoing lateral vibrations. The solution is a one-dimensional wave equation, from which natural frequencies and their associated displaced (mode) shapes can be calculated. Examples of where these vibrations may be important are mast stays, suspension bridge cables, musical instruments, and towed array sensors.

### ASSUMPTIONS

1. The strings are homogeneous and isotropic.
2. They are within the elastic limit, and obey Hooke's Law.
3. Vibration levels are 'small', so that tension does not vary with motion.
4. The tension does not vary along the length of the string.
5. The string is tensioned (i.e. not slack).
6. The flexural stiffness is very small, and can be ignored.

**THEORY**

Consider a small element of string:




Resolve forces vertically and set *"F = ma"*

$$T\left(q+\frac{\partial q}{\partial x}dx\right)-Tq=\left(rA_x dx\right)\frac{\partial^2 y}{\partial t^2}$$

therefore $$\frac{\partial q}{\partial x}=\frac{rA_x}{T}\frac{\partial^2 y}{\partial t^2}$$

Substitute $q=\frac{\partial y}{\partial x}$

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

with $$c=\sqrt{\frac{T}{rA_x}}=\text{wave speed}$$

A general solution to this equation is:

$$y\left(x,t\right)=F_1\left(ct-x\right)+F_2\left(ct+x\right)$$

Where $F_1$ and $F_2$ are arbitrary functions.

Now, the arguments of $F_1$ depend only on the value of *(ct-x)*, which includes both time and position along the string. Let us ask the question, "What conditions do we need to apply in order for the displacement, *y(x,t),* to be constant if we are only considering the $F_1$ function?" We see that this can only be achieved for all time if the value of *x* also increases, i.e., the value of *(ct-x)* is maintained constant.

This means that as time progresses, we can only have a constant displacement value if we also move in the positive x-direction. Thus, the function $F_1(ct-x)$ represents a wave traveling in the positive x-direction.

Similarly, the function $F_2(ct+x)$ represents a wave traveling in the negative x-direction.

Solving the wave equation. To solve the wave equation, we assume the motion can be separated into a time function, and a displaced shape function. Separating the variables as:

$$y(x,t) = Y(x)G(t)$$

and substituting into the previous wave equation yields:

$$\frac{\left(\dfrac{\partial^2 Y}{\partial x^2}\right)}{Y} = \frac{1}{c^2}\frac{\left(\dfrac{\partial^2 G}{\partial t^2}\right)}{G}$$

Now, the left side of this equation only depends on *x*, and the right side only depends on *t*. Since x and t are independent of each other, the equation can only be valid for all positions along the string and all time if each side of the equation evaluates to a constant. For now, let the constant be $-\left(\frac{w}{c}\right)^2$ and split the equation into 2 separate ones:

$$\frac{\partial^2 Y}{\partial x^2} + \left(\frac{w}{c}\right)^2 Y = 0$$

$$\frac{\partial^2 G}{\partial t^2} + w^2 G = 0$$

Compare these equations to the second order differential equation

$$\ddot{x} + w_n^2 x = 0$$

for which the solution is:

$$x = A.\sin(w_n t) + B.\cos(w_n t)$$

We therefore obtain a solution of the string wave equation as:

$$\boxed{\begin{aligned} &Y(x) = \left\{A.\sin\left(\frac{wx}{c}\right) + B.\cos\left(\frac{wx}{c}\right)\right\} \\ &G(t) = \sin(wt) \\ \text{hence}\quad &y(x,t) = Y(x).G(t) = \left\{A.\sin\left(\frac{wx}{c}\right) + B.\cos\left(\frac{wx}{c}\right)\right\}\sin(wt) \end{aligned}}$$

where *A* and *B* are arbitrary constants that depend on the boundary conditions at $x=0$ and $x=L$.

## BOUNDARY CONDITIONS FOR A STRING STRETCHED BETWEEN 2 FIXED POINTS

These boundary conditions are the ones most often found for tensioned strings.

*y(0,t)* = 0 yields:

$$y(x,t) = Y(x).G(t) = \left\{A.\sin\left(\frac{wx}{c}\right) + B.\cos\left(\frac{wx}{c}\right)\right\}\sin(wt)$$

$$y(0,t) = 0 = \{0 + B.\cos(0)\}$$

hence $$B = 0$$

With B = 0, then *y(L,t)* = 0 leads to

$$\sin\left(\frac{wL}{c}\right) = 0$$

hence $$\frac{w_n L}{c} = np \qquad \text{with } n = 1, 2, .....$$

This yields:

$$f_n = \frac{nc}{2L} = \frac{n}{2L}\sqrt{\frac{T}{r A_X}} \qquad n = 1, 2, 3, ....$$

Also, if $l$ = wavelength, then $c = fl$, hence

$$l = \frac{c}{f} = \frac{c}{\frac{n}{2L}\sqrt{\frac{T}{r A_X}}}$$

so

$$l = \frac{2L}{n}$$

## SUMMARY

$$\boxed{\begin{array}{l} f_n = \dfrac{nc}{2L} = \dfrac{n}{2L}\sqrt{\dfrac{T}{r A_X}} \qquad n = 1, 2, 3, .... \\ l = \dfrac{2L}{n} \end{array}}$$

**DISCUSSION**

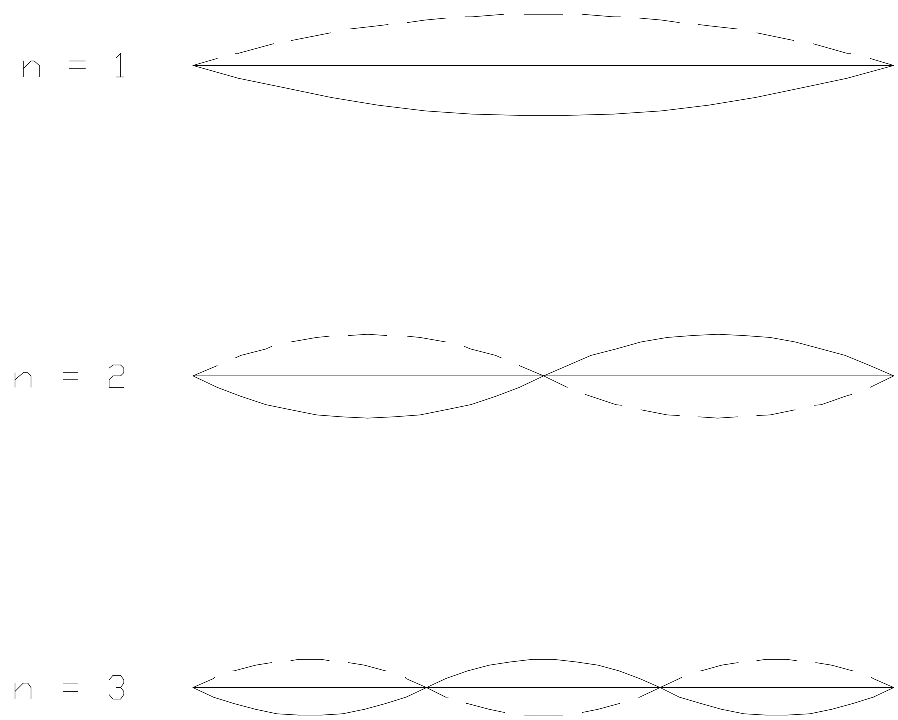
1. There are two waves of energy, one traveling in the positive x-direction, and one in the negative x-direction.

2. The total response is the product of a spatial and a time function. The time function is harmonic with respect to time. The spatial function is harmonic with respect to *x*.

3. The natural frequencies of the string are given by:

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{r A_x}} \qquad n = 1, 2, 3, ....$$

4. $f_n \propto n$




5. $f_n \propto \frac{1}{L}$

6. $f_n \propto \frac{1}{\sqrt{r A_x}}$ i.e., increase the linear density, decrease the natural frequency

Note that for <u>solid, circular</u> strings, the linear density is proportional to cross sectional area, which means $f_n$ is inversely proportional to the diameter.

**EXAMPLES**

1. A steel wire, 5 mm diameter and 15 m long, is tensioned with a force of 10 N. How much time does it take for a flexural wave to travel from one end of the wire to the other?

$$c = \sqrt{\frac{T}{r A_X}} = \sqrt{\frac{10}{7843p \frac{\left(5\times10^{-3}\right)^2}{4}}} = 8.058 m/s$$

$$\text{length} = 15m \quad \therefore \quad \text{time} = \frac{15}{8.058} = 1.86s$$

2. An 80 cm long string has a fundamental natural frequency of 440 Hz. What frequency might it vibrate at if it is lightly touched 20 cm from one end?

> There must be a node 20 cm from one end, and therefore nodes at 40 cm, and 60 cm. The wavelength is 40 cm and we are considering the 4th natural frequency (n=4):

(diagrams given in class)

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{r A_X}} \propto n$$

hence

$$f_4 = \frac{4}{1} f_1 = 4\times440 = 1760Hz$$

Alternative solution:

$$f_n = \frac{c}{l_n}$$

$$l_1 = 2L \quad \text{and} \quad l_4 = L/2$$

$$\text{so} \quad f_4 = f_1 \frac{l_1}{l_4} = 440\times\frac{2L}{L/2} = \frac{2\times440}{0.5} = 1760Hz$$

3. A ship's mast stay is 8 m long and vibrates in sympathy with ship-borne vibrations in the frequency range 5 to 10 Hz. Making suitable assumptions and calculations, what can be done to reduce the vibration?

First determine which mode of vibration is probably causing the vibration (the mast stay will vibrate in sympathy with the ship-borne vibrations). Making the following assumptions: The stay is a 3 mm diameter, solid steel wire, with a tension of 200 N (20 kg dead weight).

$$r A_X = 7843 \frac{p\left(3\times10^{-3}\right)^2}{4} \approx 0.055 kg/m$$

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{r A_X}} = \frac{1}{2\times8}\sqrt{\frac{200}{0.055}} = 3.8Hz$$

$$f_2 = 2\times f_1 = 7.5Hz$$

$$f_3 = 3\times f_1 = 11.3Hz$$

So the stay is probably vibrating in the second mode. Some ideas to think about:

Increase/decrease tension (what about the other natural frequencies)

"Snubbers"

Change the weight (diameter) of the stay

## ASSIGNMENTS – INTRODUCTION & REVIEW

1. Calculate the natural frequency (in both Hz and rad/s) of a 500 g mass suspended on a spring with a stiffness of 2.4 kN/mm.

2. A sine wave has a peak-to-peak amplitude of 3 mm and a period of 5 ms. Find:
   a) The frequency and circular frequency
   b) The peak velocity and peak acceleration.

3. Two complex vectors are $\bar{a} = \left(2\hat{i} + 3\hat{j}\right)$ and $\bar{b} = \left(6\hat{i} - \hat{j}\right)$. Find:
   a) Their sum, giving your answer as an amplitude and angle.
   b) Their cross product, giving your answer as a complex number.
   c) Their scalar (or dot) product.

4. What is the rms amplitude of a sine wave with a peak-to-peak amplitude of 6 mm?

5. An amplifier has a gain of 15. Express this in decibels.

## ASSIGNMENTS – STRINGS

1. Find the wave velocity along a rope whose mass is 0.4-kg/m, when stretched to a tension of 500-N.

2. Derive the equation for the natural frequencies of a uniform cord of length L, fixed at both ends. The cord is stretched to a tension T, and its mass per unit length is $r$.

## SOLUTIONS – INTRODUCTION & REVIEW

1. Calculate the natural frequency (in both Hz and rad/s) of a 500 g mass suspended on a spring with a stiffness of 2.4 kN/mm.

$$w_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.4\times10^6}{0.5}} = 2191 \text{ rad/s}$$

$$f_n = \frac{w_n}{2p} = \frac{2191}{2p} = 349 \text{ Hz}$$

2. A sine wave has a peak-to-peak amplitude of 3 mm and a period of 5 ms. Find:
   a) The frequency and circular frequency

$$f = \frac{1}{t} = \frac{1}{0.005} = 200 \text{ Hz}$$

$$w = 2pf = 2p\,200 = 1257 \text{ rad/s}$$

   b) The peak velocity and peak acceleration.

$$x = A.\sin(wt)$$

$$\dot{x} = wA.\cos(wt)$$

$$\ddot{x} = -w^2A.\sin(wt)$$

so

$$v_{PEAK} = \dot{x}_{PEAK} = wA = 1257\frac{0.003}{2} = 1.885 \text{ m/s}$$

$$a_{PEAK} = \ddot{x}_{PEAK} = w^2A = 1257^2\,\frac{0.003}{2} = 2369 \text{ m/s}^2$$

Note: These are mean-to-peak values.

3. Two complex vectors are $\bar{a} = \left(2\hat{i} + 3\hat{j}\right)$ and $\bar{b} = \left(6\hat{i} - \hat{j}\right)$. Find:
   a) Their sum, giving your answer as an amplitude and angle.

$$\bar{R} = \bar{a} + \bar{b} = \left(2\hat{i} + 3\hat{j}\right) + \left(6\hat{i} - \hat{j}\right) = \left(8\hat{i} + 2\hat{j}\right)$$

$$\left|\bar{R}\right| = \sqrt{8^2 + 2^2} = 8.246$$

$$j = \tan^{-1}\left(\frac{2}{8}\right) = 14.04^O$$

Answer: 8.246 at $14.04^O$ counterclockwise from the x-axis.

b) Their cross product, giving your answer as a complex number.

$$\bar{a} \times \bar{b} = \left(2\hat{i} + 3\hat{j}\right) \times \left(6\hat{i} - \hat{j}\right) = \left(-2\hat{k} - 18\hat{k}\right) = -20\hat{k}$$

c) Their scalar (or dot) product.

$$\bar{a} \bullet \bar{b} = \left(2\hat{i} + 3\hat{j}\right) \bullet \left(6\hat{i} - \hat{j}\right) = (12 - 3) = 9$$

4. What is the rms amplitude of a sine wave with a peak-to-peak amplitude of 6 mm?

$$\text{rms amplitude} = \frac{(\text{mean-to-peak amplitude})}{\sqrt{2}} = \frac{3}{\sqrt{2}} = 2.121 \text{ mm}$$

5. An amplifier has a gain of 15. Express this in decibels.

$$\text{gain in dB} = 10 \times \log_{10}(\text{power ratio}) = 20 \times \log_{10}(\text{amplitude ratio})$$
$$= 20 \times \log_{10}(15) = 23.52 \text{ dB}$$

## **SOLUTIONS – STRINGS**

1. Find the wave velocity along a rope whose mass is 0.4-kg/m, when stretched to a tension of 500-N.

$$c = \sqrt{\frac{T}{r A_X}} = \sqrt{\frac{500}{0.4}} = 35.4 \text{ m/s}$$

2. Derive the equation for the natural frequencies of a uniform cord of length L, fixed at both ends. The cord is stretched to a tension T, and its mass per unit length is $r$.

$$y(x,t) = \left(A.\sin\left(\frac{wx}{c}\right) + B.\cos\left(\frac{wx}{c}\right)\right)\sin(wt)$$

At $x = 0$ ; $y = 0$ hence $B = 0$

At $x = L$ ; $y = 0$ hence $\sin\left(\frac{w_{\mathrm{n}} L}{c}\right) = 0$

$$\left(\frac{w_{\mathrm{n}} L}{c}\right) = np \quad n = 1,\ 2,\ 3,\ \ldots\ldots$$

$$f_n = \frac{w_{\mathrm{n}}}{2p} = \frac{\mathrm{nc}}{2\mathrm{L}} = \frac{\mathrm{n}}{2\mathrm{L}}\sqrt{\frac{\mathrm{T}}{r\,\mathrm{A_X}}} \quad n = 1,\ 2,\ 3,\ \ldots\ldots$$